SIMPSON TYPE INEQUALITIES VIA $m-$ AND $(\alpha, m)-$ LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we obtain some Simpson type inequalities for functions whose derivatives in absolute value are $m-$ and $(\alpha, m)-$ logarithmically convex functions.

1. INTRODUCTION

Suppose $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_\infty = \sup \left| f^{(4)}(x) \right| < \infty$. The following inequality

$$\begin{aligned} & \left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f \left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq & \frac{1}{2880} \left\| f^{(4)} \right\|_\infty (b-a)^4 \end{aligned}$$

is well known in the literature as Simpson's inequality.

For some recent results related to Simpson's inequality see [1]-[6].

The function $f : [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Convex functions play an important role in many branches of mathematics and the other sciences as engineering, economics and optimization theory. Several extensions, generalizations and refinements have been presented by researchers.

Definition 1. *([7]) A function $f : [0, b] \to (0, \infty)$ is said to be $m-$logarithmically convex if the inequality*

$$f(tx + m(1-t)y) \leq [f(x)]^t [f(y)]^{m(1-t)} \tag{1.1}$$

holds for all $x, y \in [0, b]$, $m \in (0, 1]$, and $t \in [0, 1]$.

Obviously, if putting $m = 1$ in Definition 3, then f is just the ordinary logarithmically convex function on $[0, b]$.

Definition 2. *([7]) A function $f : [0, b] \to (0, \infty)$ is said to be $(\alpha, m)-$logarithmically convex if*

$$f(tx + m(1-t)y) \leq [f(x)]^{t^\alpha} [f(y)]^{m(1-t^\alpha)} \tag{1.2}$$

holds for all $x, y \in [0, b]$, $(\alpha, m) \in (0, 1] \times (0, 1]$, and $t \in [0, 1]$.

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Clearly, when taking $\alpha = 1$ in Definition 4, then f becomes the standard $m-$logarithmically convex function on $[0, b]$.

The main purpose of this paper is to prove some new inequalities of Simpson's type for functions whose first derivatives $m-$ and $(\alpha, m)-$logarithmically convex functions by using Lemma 1.

2. SIMPSON TYPE INEQUALITIES

We have used the following Lemmas to obtain our main results.

Lemma 1. *(See [1]) Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\begin{aligned} & \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx \\ = \quad & (b-a)\int_0^1 m(t)f'(tb+(1-t)a)dt, \end{aligned}$$

where

$$m(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right) \\ t-\frac{5}{6}, & t\in\left[\frac{1}{2},1\right] \end{cases}.$$

Theorem 1. *Let $f : [0, \infty) \to (0, \infty)$ be differentiable mapping with $a, b \in [0, \infty)$ such that $a < b$. If $|f'(x)|$ is $(\alpha, m)-$logarithmically convex function with $f' \in L[a, b]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m K_1(\alpha, m).$$

where

$$\mu=\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}, \qquad K_1(\alpha,m)=\begin{cases} \frac{5}{36} & ,\mu=1 \\ F_1(\mu,\alpha) & ,\mu<1 \end{cases}$$

and

$$F_1(\mu,\alpha)=\frac{1}{12\alpha^2\ln\mu}\left[12\mu^{\frac{\alpha}{6}}+4\alpha\mu^{\frac{\alpha}{2}}+6(\mu^\alpha-1)-\alpha\ln\mu(\mu^\alpha+1)\right].$$

Proof. From Lemma 1 and using the (α,m) −logarithmically convexity of $|f'(x)|$ we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left\{\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt\right\} \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt \\
&\left.+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{t^\alpha}dt\right\}.
\end{aligned}$$

If $\mu=1$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{5(b-a)}{36}\left|f'\left(\frac{a}{m}\right)\right|^m.
\end{aligned}$$

If $\mu<1$, then $\mu^{t^\alpha}\leq\mu^{\alpha t}$, so we can write

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt \\
&\left.+\int_{\frac{5}{6}}^{1}\left(t-\frac{5}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha t}dt\right\}.
\end{aligned}$$

By making use of the neccessary process, the proof is completed. □

Theorem 2. *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is (α,m) −logarithmically convex function with $f'\in L[a,b]$ and $(\alpha,m)\in(0,1]\times(0,1]$ for some fixed $q>1$, then the following inequality*

holds:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\left\{\begin{array}{ll}(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{\mu^{\frac{\alpha q}{2}}-1}{\alpha q\ln\mu}\right)^{\frac{1}{q}}+\left(\frac{\mu^{\alpha q}-\mu^{\frac{\alpha q}{2}}}{\alpha q\ln\mu}\right)^{\frac{1}{q}}\right] & ,\mu<1\\ (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} & ,\mu=1\end{array}\right.$$

where $p=\frac{q}{q-1}$ *and* $\mu=\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}$.

Proof. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &(b-a)\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|f'|^q$ is $(\alpha,m)-$logarithmically convex function, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left\{\left(\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)^p dt+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)^p dt\right)^{\frac{1}{p}}\right.\\ &\times\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right)^{\frac{1}{q}}\\ &+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)^p dt+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)^p dt\right)^{\frac{1}{p}}\\ &\left.\times\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right)^{\frac{1}{q}}\right\}\\ =\ &(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\\ &\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right)^{\frac{1}{q}}\right\}\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}=1$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}.
\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}<1$, then $\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^{\alpha}}\leq\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt}$, thereby

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & (b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt}dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

Theorem 3. *Let $f:[0,\infty)\to(0,\infty)$ be differentiable mapping with $a,b\in[0,\infty)$ such that $a<b$. If $|f'(x)|^q$ is $(\alpha,m)-$logarithmically convex function with $f'\in L[a,b]$ and $(\alpha,m)\in(0,1]\times(0,1]$ for some fixed $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \begin{cases}(b-a)\left|f'\left(\frac{a}{m}\right)\right|^m\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left[\left(F_1(\mu,\alpha q)\right)^{\frac{1}{q}}+\left(F_2(\mu,\alpha q)\right)^{\frac{1}{q}}\right] & ,\mu<1 \\ \frac{5(b-a)}{36}\left|f'\left(\frac{a}{m}\right)\right|^m & ,\mu=1\end{cases}
\end{aligned}$$

where

$$\begin{aligned}
F_1(\mu,\alpha q) &= \frac{1}{12(\alpha q)^2\ln\mu}\left[12\mu^{\frac{\alpha q}{6}}-6\left(1+\mu^{\frac{\alpha q}{2}}\right)+\alpha q\left(2\mu^{\frac{\alpha q}{2}}-\ln\mu\right)\right] \\
F_2(\mu,\alpha q) &= \frac{1}{12(\alpha q)^2\ln\mu}\left[\mu^{\alpha q}\left(6-\alpha q\ln\mu\right)+\mu^{\frac{\alpha q}{2}}\left(6+2\alpha q\ln\mu\right)\right].
\end{aligned}$$

Proof. From Lemma 1 and using the power-mean inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ &(b-a) \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'|^q$ is (α,m) −logarithmically convex function, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt \\
\leq\ &\left|f'\left(\frac{a}{m}\right)\right|^{qm}\left\{\int_0^{\frac{1}{6}}\left(\frac{1}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right. \\
&\left.+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(t-\frac{1}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt \\
\leq\ &\left|f'\left(\frac{a}{m}\right)\right|^{qm}\left\{\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-t\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right. \\
&\left.+\int_{\frac{5}{6}}^1\left(t-\frac{5}{6}\right)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{qt^\alpha}dt\right\}.
\end{aligned}$$

If $\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}=1$, we obtain

$$\int_0^{\frac{1}{2}}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\leq\frac{5}{72}\left|f'\left(\frac{a}{m}\right)\right|^{qm}$$

and

$$\int_{\frac{1}{2}}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\leq\frac{5}{72}\left|f'\left(\frac{a}{m}\right)\right|^{qm}.$$

If we take $\frac{|f'(b)|}{|f'(\frac{a}{m})|^m} < 1$, then $\left(\frac{|f'(b)|}{|f'(\frac{a}{m})|^m}\right)^{qt^\alpha} \leq \left(\frac{|f'(b)|}{|f'(\frac{a}{m})|^m}\right)^{\alpha qt}$, thereby

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left|t-\frac{1}{6}\right| |f'(tb+(1-t)a)|^q \, dt \\
\leq\ & \left|f'\left(\frac{a}{m}\right)\right|^{qm} \left\{ \int_0^{\frac{1}{6}} \left(\frac{1}{6}-t\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right. \\
& \left. + \int_{\frac{1}{6}}^{\frac{1}{2}} \left(t-\frac{1}{6}\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left|t-\frac{5}{6}\right| |f'(tb+(1-t)a)|^q \, dt \\
\leq\ & \left|f'\left(\frac{a}{m}\right)\right|^{qm} \left\{ \int_{\frac{1}{2}}^{\frac{5}{6}} \left(\frac{5}{6}-t\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right. \\
& \left. + \int_{\frac{5}{6}}^{1} \left(t-\frac{5}{6}\right) \left(\frac{|f'(b)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)^{\alpha qt} dt \right\}.
\end{aligned}$$

Combining all the above inequalities gives us the desired result. □

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